Dynamics of regular birational maps in $\mathbb{P}^k$

Tien-Cuong Dinh and Nessim Sibony

Abstract

We develop the study of some spaces of currents of bidegree (p,p). As an application we construct the equilibrium measure for a large class of birational maps of $\mathbb{P}^k$, as intersection of Green currents. We show that these currents are extremal and that the corresponding measure is mixing.

1 Introduction

In [17], the second author has introduced a class of polynomial automorphisms of $\mathbb{C}^k$ – *regular automorphisms* – and has constructed for such maps the equilibrium measures as intersection of invariant positif closed currents – *Green currents* (see also [16, 8]). The measure is proved to be mixing when $k=2$ or 3. Regular polynomial automorphisms are Zariski dense in the space of polynomial automorphisms of a given algebraic degree. In dimension 2, these maps are Hénon type automorphisms (see [1, 2, 12]).

In this paper, we develop the theory of some spaces of currents and we construct Green currents for a larger class of birational maps of $\mathbb{P}^k$. We show that the Green currents are extremal and we obtain a mixing measure as intersection of these currents. Every small pertubation of regular polynomial automorphisms belongs to this class. Our method can be extended to some rational non-invertible self-maps of $\mathbb{P}^k$ and to random iteration.

For a Hénon automorphism f of $\mathbb{C}^2$, it was proved in [13] that the Green current T_+ is the unique positive closed $(1,1)$-current of mass 1 supported on $K^+ := \{z,\ (f^n(z))\ \text{bounded}\}$. In particular, this current is extremal. The result was extended to regular automorphisms in [17] and to weakly regular automorphisms in [16]. Here, we deal with (p,p)-currents, $p>1$. The question is to prove their extremality which implies the mixing of the equilibrium measure.

The problem was already solved for automorphisms of compact Kähler manifolds under the natural assumption that their dynamical degrees are distinct. We proved that the Green currents are almost extremal, i.e. they belong to finite dimensional extremal faces of the cone of positive closed currents. We then constructed a mixing measure [10].

We use here the same *method of* $\mathrm{dd^c}$*-resolution* as in [7, 9, 10] to study the Green current of some birational maps of $\mathbb{P}^k$. The cohomology space is simpler, but we have to extend our calculus to deal with indeterminacy set (see also [8, 6]). Most of the paper deals with the extension of the calculus to new spaces of currents. Basically the problem is to give a meaning to the formula

$$\langle f^*(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle$$

when f has indeterminacy points (see Proposition 3.5). We believe that this can be applied in other contexts.

In [15] Guedj has independently proved, for weakly regular automorphisms of $\mathbb{C}^k$, that the Green currents of the right degrees are extremal.

We describe now our situation. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a birational map of algebraic degree $d \geq 2$. Let $I^\pm$ be the indeterminacy set of $f^{\pm 1}$.

Definition 1.1 We say that f is *regular* if there exists an integer s, $1 \leq s \leq k-1$, and open sets $V^\pm$, $U^\pm$ such that

1. $\overline{V}^\pm \cap \overline{U}^\pm = \emptyset$, $\overline{V}^\pm \subset U^\mp$ and $I^\pm \subset V^\pm$.
2. There is a smooth positive closed $(k-s, k-s)$-form Θ^+ supported in $\mathbb{P}^k \setminus \overline{V}^+$, strictly positive on $\overline{U}^+$, and a smooth positive closed (s, s)-form Θ^- supported in $\mathbb{P}^k \setminus \overline{V}^-$, strictly positive on $\overline{U}^-$.
3. f maps $\mathbb{P}^k \setminus V^+$ into U^+; f^{-1} maps $\mathbb{P}^k \setminus V^-$ into U^-.

Observe that if $\mathbb{P}^k \setminus \overline{V}^+$ (resp. $\mathbb{P}^k \setminus \overline{V}^-$) is a union of analytic subsets of dimension s (resp. $k-s$) of $\mathbb{P}^k$, it carries a form Θ^+ (resp. Θ^-) as above. If f is regular and σ_1, σ_2 are automorphisms of $\mathbb{P}^k$ close to the identity, then $\sigma_1 \circ f \circ \sigma_2$ is regular. When f is a polynomial automorphism, this definition is equivalent to the definition of [17], i.e. to the fact that $I^+ \cap I^- = \emptyset$.

Consider a regular birational map f of algebraic degree $d \geq 2$. Let δ be the algebraic degree of f^{-1}. We show that the dynamical degree d_p of f is

equal to d^p for $1 \leq p \leq s$, the dynamical degree δ_q of f^{-1} is equal to δ^q for $1 \leq q \leq k-s$ and $d^s = \delta^{k-s}$ (Proposition 3.2). We also prove that $f^{\pm 1}$ are *algebraically stable*, i.e. no hypersurface is sent under an iterate of $f^{\pm 1}$ to its indeterminacy set. Hence, we can construct for $f^{\pm 1}$ Green currents $T_\pm$ of bidegree $(1,1)$ and of mass 1. The current T_+ (resp. T_-) has Hölder continuous local potentials in $\mathbb{P}^k \setminus \overline{V}^+$ (resp. $\mathbb{P}^k \setminus \overline{V}^-$) and satisfies the relation $f^*(T_+) = dT_+$ (resp. $f_*(T_-) = \delta T_-$) in $\mathbb{P}^k$ [17].

Let $I_n^\pm$ be the indeterminacy set of $f^{\pm n}$. Define $U_\infty^+ := \cup_{n\geq 0} f^{-n}(U^+) \setminus I_n^+$ and $U_\infty^- := \cup_{n\geq 0} f^n(U^-) \setminus I_n^-$. Our main result is the following theorem.

Theorem 1.2 *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a regular birational map as above. Then for every p, q such that $1 \leq p \leq s$ and $1 \leq q \leq k-s$, the following holds.*

1. *If T is a closed positive (p,p)-current on $\mathbb{P}^k$ of mass 1 which belongs to $\mathrm{PC}_p(V^+)$, then $d^{-np} f^{n*}(T)$ converge weakly in U_∞^+ to T_+^p. If T is a closed positive (q,q)-current on $\mathbb{P}^k$ of mass 1 which belongs to $\mathrm{PC}_q(V^-)$, then $\delta^{-nq}(f^n)_*(T)$ converge weakly in U_∞^- to T_-^q.*

2. *The currents T_+^p and T_-^q are extremal in the following sense. For every positive closed (p,p)-current S such that $S \leq T_+^p$ in $\mathbb{P}^k$, we have $S = cT_+^p$ in U_∞^+ where $c := \|S\|$. Analogously for T_-^q.*

3. *The probability measure $\mu = T_+^s \wedge T_-^{k-s}$ is invariant, mixing and supported in $U^+ \cap U^-$.*

The spaces PC_p will be defined in Section 2. The operator f^* on positive closed currents will be defined in Section 3. We use the method of dd^c-resolution (see [7, 9, 10]) in order to prove a convergence result, stronger than the weak convergence (point 1 of Theorem 1.2). This will be done in Section 4. The method gives also a new construction of Green currents and implies their extremality (point 2 of Theorem 1.2). The mixing of μ is a consequence of point 2 (see [17, 16, 10] for the proof).

The spaces of currents we use as in [7, 9, 10] are probably of interest: they allow to consider intersections of currents of bidegree (p,p), $p > 1$ (see Remark 2.3).

In [5], the first named author proved that T_+^s and T_-^{k-s} are weakly laminar (see [2] for Hénon maps). The Hölder continuity of local potentials of $T_\pm$ on

$U^{\pm}$ implies that the measure μ is PC. It has positive Hausdorff dimension and has no mass on pluripolar sets (see for example [17]).

This article replaces the first version of the same paper of January 2004.

2 DSH and PC currents

We will introduce two classes of currents in $\mathbb{P}^k$. Let V be an open set in $\mathbb{P}^k$. The class $\mathrm{DSH}^\bullet(V)$ is *the space of test currents*. For the bidegree $(0,0)$, these currents are Differences of q.p.S.H. functions which are pluriharmonic in a neigbourhood of $\overline{V}$. Recall that an L^1 function $\varphi : \mathbb{P}^k \to \mathbb{R}\cup\{-\infty\}$ is *q.p.s.h.* if it is upper semi-continuous and if $dd^c\varphi \geq -c\omega$, $c>0$, in the sense of currents. Here ω is the standard Fubini-Study form on $\mathbb{P}^k$ that we normalize by $\int \omega^k = 1$. A set $E\subset \mathbb{P}^k$ is *pluripolar* if $E\subset\{\varphi=-\infty\}$ for a q.p.s.h. function φ.

The class $\mathrm{PC}_\bullet(V)$ is the space of currents of zero order satisfying some regularity property in $\mathbb{P}^k\setminus\overline{V}$. For example, such a positive closed current of bidegree $(1,1)$ has Continuous local Potentials in $\mathbb{P}^k\setminus\overline{V}$ (Proposition 2.2).

Let $\mathrm{DSH}^{k-p}(V)$ denote the space of real-valued $(k-p,k-p)$-currents $\Phi=\Phi_1-\Phi_2$ on $\mathbb{P}^k$ such that

1. Φ_i are negative, $\Phi_{i|V}$ are $\mathrm{L}^\infty_{\mathrm{loc}}$ forms on V;
2. $dd^c\Phi_i=\Omega_i^+-\Omega_i^-$ with $\Omega_i^\pm$ positive closed currents supported in $\mathbb{P}^k\setminus\overline{V}$.

The mass of a positive or negative current S of bidegree $(k-p,k-p)$ is given by the formula $\|S\| := |\int S\wedge\omega^p|$. Observe that $\|\Omega_i^+\|=\|\Omega_i^-\|$. Define

$$\|\Phi\|_{\mathrm{DSH}} := \min\big\{\|\Phi_1\|+\|\Phi_2\|+\|\Omega_1^+\|+\|\Omega_2^+\|, \Phi_i,\ \Omega_i^\pm \text{ as above}\big\}.$$

So, positive closed currents supported in $\mathbb{P}^k\setminus\overline{V}$ are elements of $\mathrm{DSH}^\bullet(V)$. If S is such a current and φ is a q.p.s.h. function integrable with respect to the trace measure of S, then $\varphi S\in\mathrm{DSH}^\bullet(V)$.

A *topology* on $\mathrm{DSH}^\bullet(V)$ is defined as follows: $\Phi^{(n)}\to\Phi$ in $\mathrm{DSH}^\bullet(V)$ if we can write $\Phi^{(n)}=\Phi_1^{(n)}-\Phi_2^{(n)}$, $dd^c\Phi_i^{(n)}=\Omega_i^{(n)+}-\Omega_i^{(n)-}$ as above and

1. $\Phi^{(n)}\to\Phi$ weakly in $\mathbb{P}^k$.
2. $(\|\Phi_i^{(n)}\|+\|\Omega_i^{(n)+}\|)_{n\geq 1}$ is bounded.

3. The $\Phi_i^{(n)}$'s are locally uniformly bounded in V.

4. The $\Omega_i^{(n)\pm}$'s are supported in the same compact subset of $\mathbb{P}^k \setminus \overline{V}$.

It is a topology associated to an inductive limit.

Observe that smooth forms in $\mathrm{DSH}^\bullet(V)$ are dense in this space. This can be checked by the standard regularization using automorphisms of $\mathbb{P}^k$.

The following proposition allows to construct currents in $\mathrm{DSH}^\bullet(V)$ as solutions of dd^c-equation and shows that they can be used as quasi-potentials of positive closed currents (see also [10]).

Proposition 2.1 *Let Θ be a smooth positive closed $(k-p+1, k-p+1)$-form of mass 1 supported in a compact $K \subset \mathbb{P}^k \setminus \overline{V}$. Let Ω be a positive closed $(k-p+1, k-p+1)$-current of mass m supported in K. Then, there exists a <u>negative</u> $(k-p, k-p)$-form $\Phi \in \mathcal{C}^\infty(\mathbb{P}^k \setminus K) \cap \mathrm{DSH}^{k-p}(V)$ with L^1 coefficients, such that $dd^c\Phi = \Omega - m\Theta$. Moreover, Φ depends linearly and continuously on Ω. We also have $\|\Phi\|_{\mathrm{L}^\infty(V)} + \|\Phi\|_{\mathrm{DSH}} \leq c_K m$ where $c_K > 0$ is a constant independent of Ω. The form Φ is continuous where Ω is continuous.*

Proof. The diagonal Δ of $\mathbb{P}^k \times \mathbb{P}^k$ is cohomologous to the positive closed form

$$\alpha(z, w) := \Theta(z) \wedge \omega^{p-1}(w) + \sum_{i \neq k-p+1} \omega^i(z) \wedge \omega^{k-i}(w).$$

Following [3, Prop. 6.2.3], since $\mathbb{P}^k \times \mathbb{P}^k$ is homogeneous, we can find a negative kernel $G(z, w)$ smooth outside Δ such that $dd^c G = [\Delta] - \alpha$ and whose coefficients are, in absolute value, smaller than $c|z - w|^{1-2k}$, $c > 0$. Define the negative L^1 form Φ by

$$\Phi(z) := \int_{w \in \mathbb{P}^k} G(z, w) \wedge \Omega(w).$$

If π_1 and π_2 denote the projections of $\mathbb{P}^k \times \mathbb{P}^k$ on its factors, we have $\Phi = (\pi_1)_*(G \wedge \pi_2^*(\Omega))$ and $dd^c\Phi = (\pi_1)_*(([\Delta] - \alpha) \wedge \pi_2^*(\Omega)) = \Omega - m\Theta$. The properties of G imply that Φ is smooth on $\mathbb{P}^k \setminus K$, depends continuously on Ω and $\|\Phi\|_{\mathrm{L}^\infty(V)} + \|\Phi\|_{\mathrm{DSH}} \leq c_K m$. It is clear that Φ is continuous where Ω is continuous. $\square$

Let $\mathrm{PC}_p(V)$ be the space of positive closed (p, p)-currents T which can be extended to a linear continuous form on $\mathrm{DSH}^{k-p}(V)$. The value of this

linear form on $\Phi \in \mathrm{DSH}^{k-p}(V)$ is denoted by $\langle T, \Phi\rangle$. Since smooth forms are dense in $\mathrm{DSH}^{k-p}(V)$ the extension is unique. Of course, if $dd^c\Phi = 0$, then $\langle T, \Phi\rangle = \int [T] \wedge [\Phi]$ where $[T]$ and $[\Phi]$ are classes of T and Φ in $H^{p,p}(X, \mathbb{C})$ and $H^{k-p,k-p}(X, \mathbb{C})$. Indeed, we can approach Φ by dd^c-closed forms in $\mathrm{DSH}^{k-p}(V)$ using automorphisms of $\mathbb{P}^k$.

The following proposition justifies our notations which suggest that currents in PC have some continuity property. Let $\mathcal{C}_{k-p+1}$ denote the cone of positive closed current Ω of bidegree $(k-p+1, k-p+1)$ supported in $\mathbb{P}^k \setminus \overline{V}$. Define a topology on $\mathcal{C}_{k-p+1}$ as follows: $\Omega_n \to \Omega$ in $\mathcal{C}_{k-p+1}$ if the Ω_n are supported in the same compact subset of $\mathbb{P}^k \setminus \overline{V}$ and $\Omega_n \to \Omega$ weakly.

Proposition 2.2 *Let $T = \alpha + dd^c U$ be a positive closed (p, p)-current, where α is a continuous (p, p)-form and U is a $(p-1, p-1)$-current on $\mathbb{P}^k$.*

1. *If the map $\Omega \mapsto \langle U, \Omega\rangle$, which is defined on smooth forms $\Omega \in \mathcal{C}_{k-p+1}$, can be extended to a continuous map on $\mathcal{C}_{k-p+1}$, then $T \in \mathrm{PC}_p(V)$. In particular, if U is a continuous form on $\mathbb{P}^k \setminus \overline{V}$, then $T \in \mathrm{PC}_p(V)$.*

2. *If $p = 1$, then $T \in \mathrm{PC}_1(V)$ if and only if T has Continuous local Potentials in $\mathbb{P}^k \setminus \overline{V}$.*

Proof. 1. Consider a test current $\Phi \in \mathrm{DSH}^{k-p}(V)$. Write $dd^c\Phi = \Omega^+ - \Omega^-$ where $\Omega^\pm \in \mathcal{C}_{k-p+1}$. When Φ and $\Omega^\pm$ are smooth, we have

$$\langle T, \Phi\rangle = \langle \alpha, \Phi\rangle + \langle U, dd^c\Phi\rangle = \langle \alpha, \Phi\rangle + \langle U, \Omega^+\rangle - \langle U, \Omega^-\rangle.$$

It is clear that if the map $\Omega \mapsto \langle U, \Omega\rangle$ is well defined and continuous on $\mathcal{C}_{k-p+1}$, then $\langle T, \Phi\rangle$ can be extended to a continuous linear form on $\mathrm{DSH}^{k-p}(V)$. Hence $T \in \mathrm{PC}_p(V)$.

Using Proposition 2.1, one can prove that the converse is also true. For this, one has only to consider V weakly $(p-1)$-convex (see the definition below) since otherwise the currents in $\mathrm{DSH}^{k-p}(V)$ are dd^c-closed.

2. We write $T = \alpha + dd^c U$ with α continuous and U a q.p.s.h. function. Let Θ be a smooth positive (k, k)-form of mass 1 supported in $\mathbb{P}^k \setminus \overline{V}$. Let $a \in \mathbb{P}^k \setminus \overline{V}$ and Φ_a be the current satisfying $dd^c\Phi_a = \delta_a - \Theta$ given by Proposition 2.1. When $T \in \mathrm{PC}_1(V)$, using a regularization of Φ_a, we get

$$\langle T, \Phi_a\rangle = \langle \alpha, \Phi_a\rangle - \langle U, \Theta\rangle + U(a).$$

Since Φ_a and $\langle T, \Phi_a\rangle$ depend continuously on a, U is continuous on $\mathbb{P}^k \setminus \overline{V}$.
$\square$

Remark 2.3 The notion of PC regularity allows to consider the intersection of currents. If T belongs to $\mathrm{PC}_p(V)$ and S be a positive closed current supported in $\mathbb{P}^k \setminus \overline{V}$, then the positive closed current $T \wedge S$ is well defined and depends continuously on S. Indeed, if φ is a test real smooth form, $\varphi \wedge S$ belongs to $\mathrm{DSH}^\bullet(V)$. So we can define $\langle T \wedge S, \varphi\rangle := \langle T, \varphi \wedge S\rangle$.

Assume now that V satisfies some convexity property. We say that V is *weakly s-convex* if there exists a non zero positive closed current Θ of bidegree $(k-s, k-s)$ supported in $\mathbb{P}^k \setminus \overline{V}$. By regularization, we can assume that Θ is smooth. Assume also that $\|\Theta\| = 1$. Observe that every positive closed current of bidegree (s, s) intersects Θ. Hence, it cannot be supported in $\overline{V}$.

Proposition 2.4 *Assume that V is weakly s-convex as above. Let $T \in \mathrm{PC}_p(V)$, $1 \leq p \leq s$. There exists $c > 0$ such that if Φ is a negative smooth $(s-p, s-p)$-form with $\mathrm{dd^c}\Phi \geq -\omega^{s-p+1}$, then $\langle T, \Phi \wedge \omega^{k-s}\rangle \geq -c(1 + \|\Phi\|)$. In particular, every q.p.s.h function is integrable with respect to the trace measure $T \wedge \omega^{k-p}$ and T has no mass on pluripolar sets.*

Proof. By scaling, we can assume that $\|\Phi\| \leq 1$. Hence, $\Phi \wedge \Theta$ belongs to a compact set of $\mathrm{DSH}^{k-p}(V)$. Since T is in $\mathrm{PC}_p(V)$, there exists $c' > 0$ independent of Φ such that $\langle T, \Phi \wedge \Theta\rangle \geq -c'$.

On the other hand, if U is a smooth negative $(k-s-1, k-s-1)$-form such that $\mathrm{dd^c}U = \Theta - \omega^{k-s}$, we have

$$
\begin{aligned}
-\int T \wedge \Phi \wedge \omega^{k-s} + \int T \wedge \Phi \wedge \Theta &= \\
= \int T \wedge \Phi \wedge \mathrm{dd^c}U = \int T \wedge \mathrm{dd^c}\Phi \wedge U &\leq -\int T \wedge \omega^{s-p+1} \wedge U.
\end{aligned}
$$

We then deduce that $\langle T, \Phi \wedge \omega^{k-s}\rangle \geq -c$ where $c > 0$ is independent of Φ.

Now consider a q.p.s.h. function φ strictly negative on $\mathbb{P}^k$ such that $\mathrm{dd^c}\varphi \geq -\omega$. Let φ_n be a sequence of negative smooth functions decreasing to φ such that $\mathrm{dd^c}\varphi_n \geq -\omega$. The first part applied to $\Phi = \varphi_n \omega^{s-p}$ gives

$$\langle T, \varphi_n \omega^{k-p}\rangle \geq -c(1 + \|\varphi_n\|_{\mathrm{L}^1}) \geq -c(1 + \|\varphi\|_{\mathrm{L}^1}).$$

It follows that $\langle T, \varphi\omega^{k-p}\rangle \geq -c(1 + \|\varphi\|_{\mathrm{L}^1})$. $\square$

The above proposition gives a version of Oka's inequality (see [14]) in the sense that T-integrability on the support of Θ implies T-integrability.

Proposition 2.5 *Let V be a weakly s-convex open set in $\mathbb{P}^k$ and $T \in \mathrm{PC}_p(V)$, $1 \leq p \leq s-1$. Let R and R_i be positive closed $(1,1)$-currents. Assume that $R = \omega + dd^c v$ and $R_i = \omega + dd^c v_i$ where v and v_i are continuous on $\mathbb{P}^k \setminus \overline{V}$. Then $R \wedge T$ is well defined and belongs to $\mathrm{PC}_{p+1}(V)$. In particular, $R_1 \wedge \ldots \wedge R_n$ is well defined and belongs to $\mathrm{PC}_n(V)$ for $1 \leq n \leq s$. If $T_i \to T$ weakly in $\mathrm{PC}_p(V)$ and $v_i \to v$ locally uniformly on $\mathbb{P}^k \setminus \overline{V}$, then $R_i \wedge T_i \to R \wedge T$ weakly in $\mathrm{PC}_{p+1}(V)$.*

Proof. We can assume that v is negative. Proposition 2.4 permits to define $R \wedge T := \omega \wedge T + dd^c(vT)$ (even without assuming that v is continuous). It is easy to check by approximation that $R \wedge T$ is positive. If $\Phi \in \mathrm{DSH}^{k-p-1}(V)$ is a smooth form, we have

$$\langle R \wedge T, \Phi \rangle := \langle T, \omega \wedge \Phi \rangle + \langle T, v dd^c \Phi \rangle.$$

When $\Phi \in \mathrm{DSH}^{k-p-1}(V)$ is not smooth, the right hand side is well defined and depends continuously on Φ (see Remark 2.3 for the definition of the measure $T \wedge dd^c\Phi$). Hence, we can extend $R \wedge T$ to a linear continuous form on $\mathrm{DSH}^{k-p-1}(V)$. It follows that $R \wedge T \in \mathrm{PC}_{p+1}(V)$. For the second part of Proposition 2.5, it follows from Proposition 2.2 that $R_1 \in \mathrm{PC}_1(V)$. We then use an induction on n.

To prove the convergence result, we use the above formula:

$$\langle R_i \wedge T_i, \Phi \rangle := \langle T_i, \omega \wedge \Phi \rangle + \langle T_i, v_i dd^c \Phi \rangle.$$

The convergence of the first term is clear for $\Phi \in \mathrm{DSH}^{k-p-1}(V)$. For the second term, observe that $T_i \wedge dd^c\Phi$ are measures with bounded mass supported in the same compact subset of $\mathbb{P}^k \setminus \overline{V}$. The convergence follows. $\square$

Let V be as in Proposition 2.5 and let A be a compact analytic subset of $\mathbb{P}^k \setminus \overline{V}$. Define $\mathcal{C}$ the cone of negative L^1 forms $\Phi \in \mathcal{C}^0(\mathbb{P}^k \setminus A) \cap \mathrm{DSH}^{k-p}(V)$ such that $dd^c\Phi = \Omega^+ - \Omega^-$ with $\Omega^\pm$ positive closed supported in $\mathbb{P}^k \setminus \overline{V}$, continuous on $\mathbb{P}^k \setminus A$ and having no mass on A. Here, $\mathcal{C}^0(\mathbb{P}^k \setminus A)$ denotes the space of continuous forms on $\mathbb{P}^k \setminus A$. We will use the following lemma in Section 4.

Lemma 2.6 *Let R_i as in Proposition 2.5. Let S be a positive closed (p,p)-current, $1 \leq p \leq s$, such that $S \leq R_1 \wedge \ldots \wedge R_p$. Then S can be extended to a continuous linear form on $\mathcal{C}$ by*

$$\langle S, \Phi \rangle := \langle S, \Phi \rangle_{\mathbb{P}^k \setminus A} := \int_{\mathbb{P}^k \setminus A} S \wedge \Phi.$$

The continuity is with respect to the topology of $\mathcal{C}^0(\mathbb{P}^k\setminus A)\cap \mathrm{DSH}^{k-p}(V)$.

Proof. Define $T_i := R_1\wedge\ldots\wedge R_i$. Let $\Phi_n\in \mathrm{DSH}^{k-p}(V)$ be smooth negative forms on $\mathbb{P}^k$ such that $\Phi_n\to\Phi$ in $\mathcal{C}^0(\mathbb{P}^k\setminus A)$ and in $\mathrm{DSH}^{k-p}(V)$. We show that $\lim\langle S,\Phi_n\rangle=\langle S,\Phi\rangle_{\mathbb{P}^k\setminus A}$. This will prove the Lemma. We have by Fatou's lemma:

$$\limsup\langle S,\Phi_n\rangle\leq\langle S,\Phi\rangle_{\mathbb{P}^k\setminus A}$$

and

$$\limsup\langle T_p-S,\Phi_n\rangle\leq\langle T_p-S,\Phi\rangle_{\mathbb{P}^k\setminus A}.$$

Since, by Proposition 2.5, $\langle T_p,\Phi_n\rangle\to\langle T_p,\Phi\rangle$, we only need to prove that $\langle T_p,\Phi\rangle=\langle T_p,\Phi\rangle_{\mathbb{P}^k\setminus A}$.

Let u be a negative q.p.s.h. function such that $dd^c u\geq-\omega$, $u=-\infty$ on A and u is smooth on $\mathbb{P}^k\setminus A$. Let χ be a smooth convex increasing function on $\mathbb{R}^-\cup\{-\infty\}$ such that $\chi(0)=1$, $\|\chi\|_{\mathcal{C}^2}\leq 4$ and $\chi=0$ on $[-\infty,-1]$. Define $u_n:=\chi(u/n)$. These functions are smooth, equal to 0 in neigbourhoods of A. We also have $dd^c u_n\geq -4n^{-1}\omega$ and $u_n\to 1$ uniformly on compact sets of $\mathbb{P}^k\setminus A$. It is sufficient to show that $\lim\langle T_p,u_n\Phi\rangle=\langle T_p,\Phi\rangle$.

Let $dd^c\Phi=\Omega^+-\Omega^-$ and define $\Omega:=\Omega^+-\Omega^-$. We have

$$\langle T_p,\Phi\rangle=\langle v_pT_{p-1},\Omega\rangle+\langle T_{p-1},\omega\wedge\Phi\rangle.$$

This is true for smooth forms and hence for Φ by approximation. On the other hand, we have

$$\langle T_p,u_n\Phi\rangle=\langle dd^c v_p\wedge T_{p-1},u_n\Phi\rangle+\langle T_{p-1},\omega\wedge u_n\Phi\rangle.$$

Using an induction on p, we only need to prove that

$$\lim\langle dd^c v_p\wedge T_{p-1},u_n\Phi\rangle=\langle v_pT_{p-1},\Omega\rangle.$$

Let $\epsilon>0$, $U\Subset\mathbb{P}^k\setminus\overline{V}$ be a neighbourhood of A, M a constant such that $M\geq-\inf_U v_p$, and $v_p^M:=\max(v_p,-M)$. Since v_p is continuous on $\mathbb{P}^k\setminus\overline{V}$, $dd^c v_p^M\wedge T_{p-1}\wedge\Phi\to dd^c v_p\wedge T_{p-1}\wedge\Phi$ on $\mathbb{P}^k\setminus A$. The measures $dd^c v_p^M\wedge T_{p-1}\wedge\Phi$ and $dd^c v_p\wedge T_{p-1}\wedge\Phi$ are equal in U. Since $u_n\to 1$ locally uniformly on $\mathbb{P}^k\setminus A$, there exists n_0 such that if $n\geq n_0$ we have

$$|\langle dd^c v_p\wedge T_{p-1},u_n\Phi\rangle-\langle dd^c v_p^M\wedge T_{p-1},u_n\Phi\rangle|\leq\epsilon.$$

Hence, if we replace v_p by $v_p^M + M$, we can assume that v_p is positive. In particular, v_p^2 is q.p.s.h. Hence, $\mathrm{dd^c}(v_p^2 T)$ is a difference of positive closed currents. It follows that $\mathrm{d}v_p$ and $\mathrm{d^c}v_p$ belong to $\mathrm{L}^2(T_{p-1})$.

We have

$$\begin{aligned}\langle \mathrm{dd^c}v_p \wedge T_{p-1}, u_n\Phi\rangle &= \langle u_n v_p T_{p-1}, \Omega\rangle - \langle \mathrm{d}u_n \wedge \mathrm{d^c}v_p \wedge T_{p-1}, \Phi\rangle + \\ &\quad + \langle \mathrm{d^c}u_n \wedge \mathrm{d}v_p \wedge T_{p-1}, \Phi\rangle - \langle \mathrm{dd^c}u_n \wedge v_p T_{p-1}, \Phi\rangle.\end{aligned}$$

By induction hypothesis, the measure $T_{p-1}\wedge\Omega$ has no mass on A (see also Remark 2.3). Hence, the first term tends to $\langle v_p T_{p-1}, \Omega\rangle$. We show that the other terms tend to 0.

Since $\pm\mathrm{dd^c}u_n \leq \mathrm{dd^c}u_n + 8n^{-1}\omega$ and $\mathrm{dd^c}u_n + 8n^{-1}\omega \geq 0$, we have:

$$\begin{aligned}|\langle \mathrm{dd^c}u_n \wedge v_p T_{p-1}, \Phi\rangle| &\lesssim -\langle \mathrm{dd^c}u_n \wedge T_{p-1} + 8n^{-1}\omega \wedge T_{p-1}, \Phi\rangle \\ &\lesssim -\langle T_{p-1}, u_n \mathrm{dd^c}\Phi\rangle - 8n^{-1}\langle T_{p-1}, \omega\wedge\Phi\rangle.\end{aligned}$$

It follows that $\langle \mathrm{dd^c}u_n \wedge v_p T_{p-1}, \Phi\rangle$ tends to 0. Indeed, since $u_n\mathrm{dd^c}\Phi \to \mathrm{dd^c}\Phi$ in $\mathrm{DSH}^{k-p+1}(V)$ and $T_{p-1} \in \mathrm{PC}_{p-1}(V)$, we have $\langle T_{p-1}, u_n\mathrm{dd^c}\Phi\rangle \to \langle T_{p-1}, \mathrm{dd^c}\Phi\rangle = 0$.

For the other terms it is sufficient to use the Cauchy-Schwarz inequality and the property that $\mathrm{d}u_n \wedge \mathrm{d^c}u_n$ can be dominated by $\mathrm{dd^c}u_n^2 + 100n^{-1}\omega$. The functions u_n^2 satisfy analogous inequalities as the u_n do. □

3 Regular birational maps

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational map of algebraic degree $d \geq 2$. In homogeneous coordinates $[z_0 : \cdots : z_k]$, we have $f = [P_0 : \cdots : P_k]$ where P_i are homogeneous polynomials of degree d without common divisor. Let Γ be the graph of f in $\mathbb{P}^k \times \mathbb{P}^k$, π_i the canonical projections of $\mathbb{P}^k \times \mathbb{P}^k$ onto its factors. If A is a subset of $\mathbb{P}^k$, define $f(A) := \pi_2\big(\pi_1^{-1}(A)\cap\Gamma\big)$ and $f^{-1}(A) := \pi_1\big(\pi_2^{-1}(A)\cap\Gamma\big)$. The operators $f_* := (\pi_2)_*(\pi_{1|\Gamma})^*$ and $f^* := (\pi_1)_*(\pi_{2|\Gamma})^*$ are well defined and continuous on L^∞ forms (forms with L^∞ coefficients) with value in spaces of L^1 forms (forms with L^1 coefficients).

We define the *dynamical degree of order* p of f by

$$\begin{aligned} d_p &:= \lim_{n\to\infty} \|f^{n*}(\omega^p)\|^{1/n} = \lim_{n\to\infty}\left(\int_{\mathbb{P}^k} f^{n*}(\omega^p)\wedge\omega^{k-p}\right)^{1/n} \\ &= \lim_{n\to\infty}\|(f^n)_*(\omega^{k-p})\|^{1/n} = \lim_{n\to\infty}\left(\int_{\mathbb{P}^k}(f^n)_*(\omega^{k-p})\wedge\omega^p\right)^{1/n} \qquad (1)\end{aligned}$$

These limits always exist [11]. It is easy to see that $d_p \leq d_1^p$. The last degree d_k is *the topological degree* of f. It is equal to $\# f^{-1}(z)$ for z generic.

Consider now, a birational map f, i.e. a map with topological degree 1. The set I^+ (resp. I^-) of points $z \in \mathbb{P}^k$ such that $f(z)$ (resp. $f^{-1}(z)$) is infinite is the *indeterminacy set* of f (resp. f^{-1}). Hence $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$ out of an analytic set. Let δ denote the algebraic degree and δ_q the dynamical degree of order q associated to f^{-1}.

Definition 3.1 We say that f is *s-regular*, $1 \leq s \leq k-1$, if there exist two open sets V, U such that

1. $\overline{V} \cap \overline{U} = \emptyset$, $I^+ \subset V$ and $I^- \subset U$.
2. There is a smooth positive closed $(k-s, k-s)$-form Θ supported in $\mathbb{P}^k \setminus \overline{V}$ and strictly positive on $\overline{U}$. We will assume that $\|\Theta\| = 1$.
3. f maps $\mathbb{P}^k \setminus V$ into U.

Observe that V is weakly s-convex. If H is a hypersurface of $\mathbb{P}^k$, then $H \not\subset \overline{V}$. It follows that H cannot be sent by an iterate of f to I^+. Hence, f is algebraically stable, i.e. $\deg(f^n) = d^n$ [17].

Proposition 3.2 *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an s-regular birational map as in Definition 3.1. Let $I_n^\pm$ be the indeterminacy set of $f^{\pm n}$. Then $I_n^+ \subset V$, $I_n^- \subset U$, $\dim I_n^+ \leq k-s-1$ and $d_p = d^p$ for $1 \leq p \leq s$. We have $(f^n)^* = (f^*)^n$ on $H^{p,p}(X, \mathbb{C})$ for $1 \leq p \leq s$. If f is regular as in Definition 1.1, then $\dim I_n^- \leq s-1$, $\delta_q = \delta^q$ for $1 \leq q \leq k-s$ and $d^s = \delta^{k-s}$.*

Proof. Since f^n is holomorphic on a neighbourhood of $\mathbb{P}^k \setminus V$, we have $I_n^+ \subset V$. Since $f^{-1} : \mathbb{P}^k \setminus U \to V$ is holomorphic, we have $I_n^- \subset U$. If $\dim I_n^+ \geq k-s$, then the current of integration on I_n^+ intersects Θ which is cohomologous to ω^{k-s} (recall $\dim H^{p,p}(\mathbb{P}^k, \mathbb{C}) = 1$). This is impossible since $I_n^+ \subset V$ and $\mathrm{supp}(\Theta) \cap V = \emptyset$.

Since f is algebraically stable, $f^{n*}(\omega)$ is a positive closed $(1,1)$-current of mass d^n and smooth on $\mathbb{P}^k \setminus I_n^+$. We have seen that $\dim I_n^+ \leq k-s-1$. The intersection theory [4, 14] implies that $f^{n*}(\omega) \wedge \ldots \wedge f^{n*}(\omega)$ (p times, $p \leq s$) is well defined and does not charge algebraic sets. It's mass is equal to d^{np}. We deduce from (1) that $d_p = d^p$ and $(f^n)^* = (f^*)^n$ on $H^{p,p}(\mathbb{P}^k, \mathbb{C})$.

When f is regular, we prove in the same way that $\dim I_n^- \leq s-1$ and $\delta_q = \delta^q$. We obtain from (1) that $d_s = \delta_{k-s}$. It follows that $d^s = \delta^{k-s}$. □

Remark 3.3 The identity $(f^n)^* = (f^*)^n$ on $H^{p,p}(X, \mathbb{C})$ corresponds to an algebraic stability of higher order. The notion can be introduced for meromorphic maps on a compact Kähler manifold. Proposition 3.2 is valid in a more general case.

Let T be a positive closed (p,p)-current on $\mathbb{P}^k$. The restriction f_0 of f to $\mathbb{P}^k \setminus f^{-1}(I^-) \cup I^+$ is an injective holomorphic map. We can define $f_0^*(T)$ on $\mathbb{P}^k \setminus f^{-1}(I^-) \cup I^+$. By approximation, one can check that this is a positive closed current of finite mass (see also [11]). Let $f^\star(T)$ denote the trivial extension of $f_0^*(T)$ on $\mathbb{P}^k$. By a theorem of Skoda [18], $f^\star(T)$ is positive and closed.

If $T_n \to T$, we have $f^\star(T_n) \to f^\star(T)$ on $\mathbb{P}^k \setminus f^{-1}(I^-) \cup I^+$. Moreover, $f^\star(T)$ is smaller than every limit value τ of the sequence $f^\star(T_n)$. More precisely, the current $\tau - f^\star(T)$ is positive closed and supported in $f^{-1}(I^-) \cup I^+$.

Assume now that $1 \leq p \leq s$. Proposition 3.2 implies that $\|f^*(T)\| = d^p\|T\|$ for T smooth. Using a regularization of T, we deduce from the above properties that $\|f^\star(T)\| \leq d^p\|T\|$. When $\|f^\star(T)\| = d^p\|T\|$, we define $f^*(T) := f^\star(T)$. We define similarly $f_\star$ and f_* on positive closed currents.

Lemma 3.4 *The operator f^* is continuous: if $f^*(T_n)$ and $f^*(T)$ are well defined in the above sense and if $T_n \to T$ then $f^*(T_n) \to f^*(T)$. If $f^*(T)$ is well defined, then so is $f^*(S)$ for every positive closed current S such that $S \leq T$.*

Proof. We have $\lim \|T_n\| = \|T\|$. It follows that $\lim \|f^*(T_n)\| = d^p\|T\| = \|f^*(T)\|$. On the other hand, $f^*(T_n) \to f^*(T)$ in $\mathbb{P}^k \setminus f^{-1}(I^-) \cup I^+$ and $f^*(T)$ does not charge $f^{-1}(I^-) \cup I^+$. Hence $f^*(T_n) \to f^*(T)$ in $\mathbb{P}^k$.

We have $\|f^\star(S)\| \leq d^p\|S\|$, $\|f^\star(T-S)\| \leq d^p\|T-S\|$ and $\|f^\star(T)\| = d^p\|T\|$. It follows that $\|f^\star(S)\| = d^p\|S\|$. Hence $f^*(S)$ is well defined. $\square$

Proposition 3.5 *The operators $f_* : \mathrm{DSH}^{k-p}(V) \to \mathrm{DSH}^{k-p}(V)$ and $f^* : \mathrm{PC}_p(V) \to \mathrm{PC}_p(V)$, $1 \leq p \leq s$, are well defined and are continuous. We have $(f^n)^* = (f^*)^n$, $\|f^*(T)\| = d^p\|T\|$ and $\langle f^*(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle$ for $T \in \mathrm{PC}_p(V)$ and $\Phi \in \mathrm{DSH}^{k-p}(V)$.*

Proof. Let $\Phi \in \mathrm{DSH}^{k-p}(V)$. Using a partition of unity, we can write $\Phi = \Phi^{(1)} + \Phi^{(2)}$ where $\Phi^{(1)}$ is a L^∞ form with compact support in V and $\Phi^{(2)}$ is a current with support in $\mathbb{P}^k \setminus I^+$. By Definition 3.1, $f^{-1} : \mathbb{P}^k \setminus \overline{U} \to V$ and $f : \mathbb{P}^k \setminus I^+ \to \mathbb{P}^k$ are holomorphic. Then $f_*(\Phi^{(1)}) = (f^{-1})^*(\Phi^{(1)})$ and

$f_*(\Phi^{(2)})$ are well defined. The first assertion follows, even $\Phi^{(1)}$ and $\Phi^{(2)}$ are not necessarily in $\mathrm{DSH}^{k-p}(V)$.

Consider now a smooth positive closed form $\Phi \in \mathrm{DSH}^{k-p}(V)$. Recall that by Proposition 2.4, if T is in $\mathrm{PC}_p(V)$, then T and $f^\star(T)$ do not charge analytic sets. We have

$$\langle f^\star(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle_{\mathbb{P}^k\setminus I^-} := \int_{\mathbb{P}^k\setminus I^-} T \wedge f_*(\Phi).$$

We next show that $\langle T, f_*(\Phi)\rangle_{\mathbb{P}^k\setminus I^-} = \langle T, f_*(\Phi)\rangle$. Let W be a form, smooth outside I^-, such that $dd^c W = f_*(\Phi) - m\omega^{k-p}$ and θ be a smooth function supported in U, equal to 1 in a neighbourhood of I^-. Here, m is the mass of $f_*(\Phi)$. Define $\Psi := dd^c(\theta W) + c\omega^{s-p}\wedge\Theta$, $c > 0$ big enough. Then Ψ is positive closed, $\mathrm{supp}(\Psi) \subset \mathbb{P}^k \setminus \overline{V}$ and $\Psi - f_*(\Phi)$ is smooth. This form Ψ belongs to $\mathrm{DSH}^{k-p}(V)$. We only need to show that $\langle T, \Psi\rangle_{\mathbb{P}^k\setminus I^-} = \langle T, \Psi\rangle$.

Let u_n as in Lemma 2.6 but we replace A by I^-. We have

$$\langle T, \Psi\rangle_{\mathbb{P}^k\setminus I^-} = \lim\langle T, u_n\Psi\rangle = \langle T, \Psi\rangle$$

because $T \in \mathrm{PC}_p(V)$ and $u_n\Psi \to \Psi$ in $\mathrm{DSH}^{k-p}(V)$. So $\langle f^\star(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle$ for $\Phi \in \mathrm{DSH}^{k-p}(V)$ smooth positive and closed.

For $\Phi = \omega^{s-p}\wedge\Theta$, we get

$$\|f^\star(T)\| = \langle f^\star(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle = d^p\|T\|.$$

The last equality follows from a regularization of the positive closed current $f_*(\Phi)$ and the properties: $\|f_*(\Phi)\| = d^p$ and $T \in \mathrm{PC}_p(V)$. Hence $f^*(T)$ is well defined and equal to $f^\star(T)$.

Assume now that Φ is a smooth positive form in $\mathrm{DSH}^{k-p}(V)$ non necessarily closed. Using a regularization of $f_*(\Phi)$ in $\mathrm{DSH}^{k-p}(V)$, we get

$$\langle f^*(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle_{\mathbb{P}^k\setminus I^-} \leq \langle T, f_*(\Phi)\rangle.$$

On the other hand, if $\Phi' \geq \Phi$ is a smooth closed form, we also have

$$\langle f^*(T), \Phi' - \Phi\rangle = \langle T, f_*(\Phi' - \Phi)\rangle_{\mathbb{P}^k\setminus I^-} \leq \langle T, f_*(\Phi' - \Phi)\rangle.$$

The equality $\langle f^*(T), \Phi'\rangle = \langle T, f_*(\Phi')\rangle$ implies that $\langle f^*(T), \Phi\rangle = \langle T, f_*(\Phi)\rangle$. This also holds for Φ smooth non positive because we can write Φ as a difference of positive forms. From the first assertion of the Proposition, it follows that the right hand side of the last equality is well defined for every $\Phi \in \mathrm{DSH}^{k-p}(V)$ and depends continuously on Φ. This allows to extend $f^*(T)$ to a continuous linear form on $\mathrm{DSH}^{k-p}(V)$. Hence $f^*(T) \in \mathrm{PC}_p(V)$. The continuity of f^* and the equality $(f^n)^* = (f^*)^n$ are clear. $\square$

4 Convergence toward the Green currents

Let f be an s-regular birational map of algebraic degree $d \geq 2$ as in Definition 3.1. Recall that the Green $(1,1)$-current $T_+ := \lim d^{-n}(f^n)^*(\omega)$ of f has continuous local potentials in a neigbourhood of $\mathbb{P}^k \setminus V$ [17]. Proposition 2.5 shows that T_+^p is well defined for $1 \leq p \leq s$. It belongs to $\mathrm{PC}_p(V)$. Moreover, we have $\lim d^{-np} f^{n*}(\omega^p) = T_+^p$ in $\mathbb{P}^k \setminus \overline{V}$. The last property follows from a uniform convergence of potentials of $d^{-n}f^{n*}(\omega)$ (see [17]). This is also reproved in Theorem 4.1. We have $f^*(T_+^p) = d^p T_+^p$.

Let I_n^+ be the indeterminacy set of f^n. Define $U_\infty := \cup_{n\geq 0} f^{-n}(U) \setminus I_n^+$. In this section, we prove the following result which implies Theorem 1.2.

Theorem 4.1 *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an s-regular birational map as above. Then for every p, $1 \leq p \leq s$, the following holds.*

1. *If $T \in \mathrm{PC}_p(V)$ is a positive closed current of mass 1, then $d^{-pn}f^{n*}(T)$ converges in U_∞ to T_+^p. Moreover, every limit value of the sequence $d^{-np}f^{n*}(T)$ is in $\mathrm{PC}_p(V)$. The convergence is valid in the weak topology of $\mathrm{PC}_p(V)$.*

2. *If S is a positive closed (p,p)-current such that $S \leq T_+^p$ in $\mathbb{P}^k$, then $S = cT_+^p$ in U_∞ where $c := \|S\|$.*

Proof. 1. Let Φ be a $(k-p, k-p)$-current in $\mathrm{DSH}^{k-p}(V)$. Write $dd^c\Phi = \Omega = \Omega^+ - \Omega^-$ where $\Omega^\pm$ are positive closed $(k-p+1, k-p+1)$-currents supported in $\mathbb{P}^k \setminus \overline{V}$. Assume that $\|\Omega^\pm\| = 1$. Define $\Omega_n^\pm := (f^n)_*(\Omega^\pm)$ and $\Omega_n = \Omega_n^+ - \Omega_n^-$ for $n \geq 0$. They are supported in U for $n \geq 1$ and we have $\|\Omega_n^\pm\| = d^{(p-1)n}$.

Let $\Phi_n^\pm$ be the solution of the equation $dd^c\Phi_n^\pm = \Omega_n^\pm - d^{(p-1)n}\omega^{s-p+1} \wedge \Theta$ given in Proposition 2.1. The $\Phi_n^\pm$'s are negative $(k-p, k-p)$-forms, smooth on V and they satisfy $\|\Phi_n^\pm\|_{\mathrm{L}^\infty(V)} + \|\Phi_n^\pm\|_{\mathrm{DSH}} \lesssim d^{(p-1)n}$. Define $\Phi_n := \Phi_n^+ - \Phi_n^-$, $\Psi_0 := \Phi - \Phi_0$ and $\Psi_{n+1} := f_*(\Phi_n) - \Phi_{n+1}$. The forms Φ_n are smooth on V, $dd^c\Phi_n = \Omega_n$ and $\|\Phi_n\|_{\mathrm{DSH}} \lesssim d^{(p-1)n}$ for $n \geq 1$. By Proposition 3.5, $\|\Psi_n\|_{\mathrm{DSH}} \lesssim d^{(p-1)n}$. Since $dd^c\Psi_n = 0$, we can associate to Ψ_n a class b_n in $H^{k-p,k-p}(\mathbb{P}^k, \mathbb{C})$. We have $\|b_n\| \lesssim \|\Psi_n\|_{\mathrm{L}^1} \lesssim d^{(p-1)n}$.

Since we assume that $T \in \mathrm{PC}_p(V)$, Proposition 3.5 allows the following

calculus

$$\begin{aligned}\langle f^{n*}(T),\Phi\rangle &= \langle f^{n*}(T),\Psi_0\rangle+\langle f^{n*}(T),\Phi_0\rangle\\ &= \langle f^{n*}(T),\Psi_0\rangle+\langle f^{(n-1)*}(T),f_*(\Phi_0)\rangle\\ &= \langle f^{n*}(T),\Psi_0\rangle+\langle f^{(n-1)*}(T),\Psi_1\rangle+\langle f^{(n-1)*}(T),\Phi_1\rangle\\ &= \langle f^{n*}(T),\Psi_0\rangle+\langle f^{(n-1)*}(T),\Psi_1\rangle+\langle f^{(n-2)*}(T),f_*(\Phi_1)\rangle.\end{aligned}$$

Using the equality $f_*(\Phi_n)=\Psi_{n+1}+\Phi_{n+1}$ we obtain by induction that

$$\begin{aligned}\langle f^{n*}(T),\Phi\rangle &= \langle f^{n*}(T),\Psi_0\rangle+\langle f^{(n-1)*}(T),\Psi_1\rangle+\cdots\\ &\quad\cdots+\langle T,\Psi_n\rangle+\langle T,\Phi_n\rangle.\end{aligned}\tag{2}$$

Since $f^{n*}(T)$ is cohomologous to $d^{pn}\omega^p$, using a regularization of $\mathrm{dd^c}$-closed currents Ψ_i in $\mathrm{DSH}^{k-p}(V)$, we get

$$\langle d^{-pn}f^{n*}(T),\Phi\rangle=\int[\omega^p]\wedge(b_0+d^{-p}b_1+\cdots+d^{-pn}b_n)+d^{-pn}\langle T,\Phi_n\rangle.$$

Recall that $T\in\mathrm{PC}_p(V)$ and $\|\Phi_n^\pm\|_{\mathrm{L}^\infty(V)}+\|\Phi_n^\pm\|_{\mathrm{DSH}}\lesssim d^{(p-1)n}$. It follows that $\lim d^{-pn}\langle T,\Phi_n\rangle=0$. The relations $\|b_n\|\lesssim d^{(p-1)n}$ imply that

$$\lim\langle d^{-pn}f^{n*}(T),\Phi\rangle=\int[\omega^p]\wedge c_\Phi\quad\text{where}\quad c_\Phi:=\sum_{n\geq 0}d^{-pn}b_n.\tag{3}$$

Propositions 2.1 and 3.5 imply also that c_Φ depends continuously on $\Phi\in\mathrm{DSH}^{k-p}(V)$. So, (3) implies that every limit value of the sequence $d^{-pn}f^{n*}(T)$ belongs to $\mathrm{PC}_p(V)$.

Consider now a smooth real-valued $(k-p,k-p)$-form Φ supported in U. Observe that Φ can be written as a difference $\Phi_1-\Phi_2$ of negative forms supported in U and that $\mathrm{dd^c}\Phi_i+c(\omega^{s-p+1}\wedge\Theta)$ is positive for $c>0$ big enough. It follows that $\Phi\in\mathrm{DSH}^{k-p}(V)$. By (3), $d^{-pn}f^{n*}(T)$ converges on U to a current which does not depend on T. Hence, $\lim d^{-pn}f^{n*}(T)=T_+^p$ on U since this is true for $T=\omega^p$ (and for $T=T_+^p$). The relation $f^{n*}(T_+^p)=d^{np}T_+^p$ implies that $\lim d^{-pn}f^{n*}(T)=T_+^p$ on U_∞.

2. Let c be the mass of S and define $S_n:=d^{np}(f^n)_\star(S)$. We have $S_n\leq T_+^p$. By Lemma 3.4, $f^{n*}(S_n)$ is well defined. From Proposition 2.4, T_+^p has no mass on analytic sets. It follows that $f^{n*}(S_n)=d^{np}S$ since this holds out of an analytic set. We also deduce that $\|S_n\|=c$.

Assume that Φ is smooth and supported in U. Proposition 2.1 shows that Φ_j and Ψ_j belong to the class $\mathcal{C}$ as in Lemma 2.6 for $A = \cup_{i\leq n} f^i(I^-)$. Hence, we can apply Lemma 2.6 to $R_i = T_+$ and to $(f^{n-j})^*(S_n)$. We get

$$\langle (f^{n-j+1})^*(S_n), \Psi_j\rangle = \langle (f^{n-j})^*, f_*(\Psi_j)\rangle$$

since these integrals can be computed out of the singularities of f, Ψ_j and $f_*(\Psi_j)$. We can then apply (2) to $S_n - cT_+^p$. Since $S_n - cT_+^p$ is cohomologous to 0, we get

$$\begin{aligned} d^{np}\langle S - cT_+^p, \Phi\rangle &= \langle f^{n*}(S_n - cT_+^p), \Phi\rangle = \langle S_n - cT_+^p, \Phi_n\rangle \\ &= \langle S_n - cT_+^p, \Phi_n^+ - \Phi_n^-\rangle. \end{aligned}$$

The relations $S_n \leq T_+^p$ and $\Phi_n^\pm \leq 0$ imply that the last expression is dominated by a combination of $\langle T_+^p, \Phi_n^+\rangle$ and of $\langle T_+^p, \Phi_n^-\rangle$. Hence, since $T_+^p \in \mathrm{PC}_p(V)$ and $d^{-(p-1)n}\Phi_n^\pm$ belong to a compact set in $\mathrm{DSH}^{k-p}(V)$, we have

$$d^{np}|\langle S - cT_+^p, \Phi\rangle| \lesssim d^{(p-1)n}.$$

It follows that $\langle S - cT_+^p, \Phi\rangle = 0$ for every smooth form Φ supported in U. Hence, $S = cT_+^p$ on U. In the same way, we show that $S_n = cT_+^p$ on U. The relations $f^{n*}(S_n) = d^{np}S$ and $f^{n*}(T_+^p) = d^{np}T_+^p$ imply that $S = cT_+^p$ on $f^{-n}(U) \setminus I_n^+$ for every $n \geq 1$. $\square$

Remark 4.2 The convergence in Theorem 3.1 is uniform on $T \in \mathrm{PC}_p(V)$ such that $|\langle T, \Phi\rangle| \leq c(\|\Phi\|_{\mathrm{L}^\infty(V)} + \|\Phi\|_{\mathrm{DSH}})$, $c > 0$, for every $\Phi \in \mathrm{DSH}^{k-p}(V)$.

References

[1] Bedford E., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$ III, *Math. Ann.*, **294** (1992), 395-420.

[2] Bedford E., Lyubich M., Smillie J., Polynomial diffeomorphisms of $\mathbb{C}^2$ IV, *Invent. Math.*, **112(1)** (1993), 77-125.

[3] Bost J.B., Gillet H., Soulé C., Heights of projective varieties and positive Green forms, *J. Amer. Math. Soc.*, **7** (1994), no. 4, 903–1027.

[4] Demailly J.P., Regularization of closed positive currents and intersection theory, *J. Algebraic Geometry*, **1** (1992), 361-409.

[5] Dinh T.C., Suites d'applications méromorphes multivaluées et courants laminaires, *preprint* (2003). arxiv.org/abs/math.DS/0309421.

[6] Dinh T.C., Dujardin R., Sibony N., On the dynamics near infinity of some polynomial mappings in $\mathbb{C}^2$, *preprint* (2004). arxiv.org/abs/math.DS/0407451.

[7] Dinh T.C., Sibony N., Dynamique des applications d'allure polynomiale, *Journal de Math. Pures et Appl.*, **82** (2003), 367-423.

[8] Dinh T.C., Sibony N., Dynamique des applications semi-régulières, *Arkiv för Mat.*, **42** (2004), 61-85.

[9] Dinh T.C., Sibony N., Distribution des valeurs de transformations méromorphes et applications, *preprint* (2003). arxiv.org/abs/math.DS/0306095.

[10] Dinh T.C., Sibony N., Green currents for automorphisms of compact Kähler manifolds, *preprint* (2003). arxiv.org/abs/math.DS/0311322.

[11] Dinh T.C., Sibony N., Une borne supérieure pour l'entropie topologique d'une application rationnelle, *Ann. of Math.*, to appear.

[12] Fornæss J.E., Sibony N., Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains, *Duke Math. J.*, **65** (1992), 345-380.

[13] Fornæss J.E., Sibony N., Complex dynamics in higher dimension, *in Complex potential theory*, (Montréal, PQ, 1993), Nato ASI series Math. and Phys. Sci., vol. **C439**, Kluwer (1994), 131-186.

[14] Fornæss J.E., Sibony N., Oka's inequality for currents and applications, *Math. Ann.*, **301** (1995), 399-419.

[15] Guedj V., Courants extrémaux et dynamique complexe, *prépublication*.

[16] Guedj V., Sibony N., Dynamics of polynomial automorphisms of $\mathbb{C}^k$, *Arkiv för Mat.*, **40** (2002), 207-243.

[17] Sibony N., Dynamique des applications rationnelles de $\mathbb{P}^k$, *Panoramas et Synthèses*, **8** (1999), 97-185.

[18] Skoda H., Prolongement des courants positifs, fermés de masse finie, *Invent. Math.*, **66** (1982), 361-376.

Tien-Cuong Dinh and Nessim Sibony, Mathématique - Bât. 425, UMR 8628, Université Paris-Sud, 91405 Orsay, France.
E-mails: (TienCuong.Dinh, Nessim.Sibony)@math.u-psud.fr